Mechanical response of linear viscoelastic composite laminates incorporating non-isothermal physical aging effects

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Abstract

This paper presents a method of predicting the mechanical response of composite laminates including the effects of linear viscoelasticity and physical aging. Effective-time theory has been used to characterize the physical aging behavior of each linear viscoelastic lamina. In accordance with experimental findings, the aging behavior of each lamina is allowed to differ in the shear and transverse directions. The mechanical loading is restricted to the linear range, which decouples the aging and load behavior. A recursive algorithm has been used to solve the hereditary convolution integral that governs the response of each ply. Classical thin-laminate theory is then used to assemble the individual ply response equations and determine the overall laminate response to general in-plane force and moment loading. The method automatically recovers the ply-level stresses and strains, which are often critical to strength and durability predictions. The model can use either lamina compliance or modulus properties as its basis. Several illustrative examples of long-term laminate response to variable loading are presented and the impact of physical aging is explored. It is shown that for multidirectional laminates, the stiffness of the lamina in the fiber directions can allow simplifications of the model. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction and background

Fiber-reinforced polymer-matrix composites (PMCs) represent an important group of materials in many modern applications, offering advantages such as improved strength-to-weight ratio, improved stiffness-to-weight ratio, property tailoring and chemical/environmental resistance. However, material behavior is more complicated than a homogenous structure on a number of levels including fabrication, design and analysis. Over the past several decades, the behavior of elastic PMCs has been analyzed using classical thin-laminate theory (CTLT) as the fundamental basis. This approach determines the in-plane mechanical response of a laminate based upon the properties of the layers (laminae) which comprise the overall laminate. CTLT does not account for many of the complicating effects of PMCs (such as edge delamination) that require three-dimensional analysis to characterize fully. Numerous texts provide an in-depth discussion of these issues for elastic PMCs [1–3].

As the operating temperature approaches the glass-transition temperature, \( T_g \), of the polymer matrix, the behavior of PMCs is complicated by the introduction of time-dependent response. Although the nature of an elastic composite can change with time (owing to degradation, moisture absorption, etc.), these considerations can often be accounted for in the above elastic approach by assuming quasi-static conditions (i.e. the use of elastic material behavior evaluated at the moment under consideration). At elevated temperatures (relative to \( T_g \)), this approach is inadequate as the material exhibits significant time-dependence as a consequence of several factors: the overall viscoelastic nature of a polymer close to \( T_g \), changes in the chemical makeup of the polymer (chemical aging), evolution towards the equilibrium state (physical aging), moisture absorption (hygrothermal effects), and evolving material defects due to loading or environment (damage).

Over the past three decades, a number of researchers have modeled the time-dependent response of PMCs by using several approaches including linear viscoelasticity, non-linear viscoelasticity and non-linear viscoplasticity. Schapery [4–6] considered both linear and non-linear
anisotropic behavior; this investigation resulted in the non-linear Schapery model which is often used as a basis for composite response prediction. Several researchers have performed laminate solutions in an incremental fashion (assuming the driving parameter is constant during each solution step) to simplify the hereditary convolution integral. Flagg and Crossman [7] studied and predicted the effects of temperature and moisture on the warpage of linear viscoelastic composite laminates. Harper and Weitsman [8] considered the effect of a variable humidity history on the warpage of antisymmetric cross-ply laminates. Brinson and coworkers [9–11] predicted the mechanical response for non-linear viscoelastic materials and demonstrated good agreement with experimental data under limited stress histories (i.e., single load/unload).

Subsequent studies have focused upon improved methods of evaluating the hereditary convolution integral that governs the mechanical response of both the linear and nonlinear (Schapery) viscoelastic composite laminae. These rely on recursive solutions of the hereditary convolution integral [12] which then permit easy implementation into a finite element method. Lin and Hwang [13] performed a linear viscoelastic finite element study of laminates under varying stress and temperature histories. Ha and Springer [14] developed a material model that combined nonlinear elasticity, viscoelasticity, and viscoplasticity. Yi and Hilton [15,16] considered delamination of linear viscoelastic laminates by studying response in the plane perpendicular to the laminae, solving the hereditary integral in the Laplace domain. In addition to the plane solutions above, two three-dimensional finite element approaches have been presented. Kennedy and Wang [17] developed such a method for nonlinear viscoelasticity incorporating the Schapery model. Most recently, Zocher et al. [18] developed a similar method that applies to linear viscoelastic non-aging materials with variable temperature histories.

While the above review demonstrates that several complete and general treatments of viscoelastic composite laminate response have been performed, few solutions have been extended to predict the effects of ongoing physical aging. Physical aging is a phenomenon that occurs in polymeric materials cooled to temperatures $T < T_m$ which causes the material to achieve thermodynamic equilibrium only after a finite (often extremely long) period of time. One effect of physical aging is that material properties (such as specific volume, compliance, modulus) change continuously. Physical aging differs from degradative phenomena (such as chemical aging and damage) because it is thermoreversible: when heated above $T_m$ for a nominal rejuvenation period, aging materials “forget” past aging histories and behave upon a subsequent quench as they did previously. While physical aging manifests itself in several ways (volumetric response, electrical properties), this paper focuses on the effects of physical aging via changes in the mechanical response (compliance, modulus).

For the composite solution methods above which are linear viscoelastic and thermorheologically simple, these effects can be included by modifying the effective (or reduced) time function in the analysis. Since the effects of physical aging in composite laminates have been experimentally observed to vary in the shear and transverse lamina directions [19–23], the effective time function needs to be directionally dependent for the most general solution (a feature generally not included in the derivations above). For the composite solutions methods using nonlinear material models, inclusion of physical aging effects is more problematic; two of the issues that must be addressed are the degree to which aging has already been incorporated into determining the non-linear response of the material (which can itself be viewed as a nonlinear response) and what effect large load levels (the typical impetus of nonlinear material behavior) have upon the state of physical aging of the polymer matrix.

For the reasons above, it is beneficial to derive a simplified composite laminate constitutive model that accounts for ongoing physical aging in a linear viscoelasticity framework. Several studies have already considered mechanical response of such systems. Brinson and Gates [24] used experimentally determined aging compliance behavior and several simplifying assumptions to predict laminate response under constant load. This approach was extended by Monaghan et al. [25] for the case of a multiple step load history. Pasricha et al. [26,27] added the effects of physical aging to the nonlinear viscoelasticity (Schapery model) solution program (BONVICA) via a change in the effective time; this approach adequately predicted long-term compliance results although concerns about the behavior of physical aging in nonlinear systems remain. BONVICA is limited to predicting either creep or stress-relaxation under either constant or cyclic loading conditions. In all of the above approaches, the aging behavior was assumed identical in both the shear and transverse lamina directions and nonisothermal aging effects were not considered.

The model presented here begins with the characterization of the compliance of a lamina. The lamina is assumed thermorheologically simple, which reduces the analysis to linear viscoelasticity when the problem is mapped to the appropriate effective time domain. This decouples the aging and mechanical response, since the aging shift factors (the basis for the effective time) can be determined from the temperature history alone (without regard to the loading applied). This greatly simplifies development of both the nonisothermal physical aging model and the mechanical response model by allowing them to be considered separately.

The Taylor method [12] is then employed to reduce the system of convolution integral equations that govern
the lamina stress-strain relationship to a system of algebraic equations in terms of the unknown stress or strain at the current time step. This result is then adjusted to account for hygrothermal strain effects. Classical thin-laminate theory employs the resulting equations to create an algebraic relationship between the laminate force/moment vector and the laminate displacement/curvature vector; hence, either can be solved for when the other is known. Once the laminate displacement vector (mid-plane extension and curvature) is established, the information from the previous solution step can be used to recover the ply-level stresses and strains at the current step. This information in turn updates the solution method for the next step. This approach is demonstrated for several examples of interest and issues regarding the impact of complete versus approximate aging algorithms are explored.

2. Momentary response for thermo-rheologically simple materials

For thermo-rheologically simple (TRS) materials, Struik [28] demonstrated that physical aging (both isothermal and nonisothermal) can be accounted for by a shift factor relating the momentary aged response (S) after an aging history to a momentary reference response curve (Sref) by the equation

\[ S(t_e, T) = S_{\text{ref}}(a_t, \hat{a}_T)|_{t_\text{ref}, T_{\text{ref}}} \]

where \( t_{\text{ref}} \) and \( T_{\text{ref}} \) are the isothermal aging time and temperature, respectively, at which the reference curve was defined, \( t_e \) and \( T \) are the aging time and temperature, respectively, at which the short-term test is taking place, and \( a_t \) and \( a_T \) are the horizontal shift factors due to aging and temperature effects, respectively. Typically, the above relationship is used to describe mechanical response functions (compliance, modulus) immediately after a step loading. The momentary nature of the response \( S \) requires that the time range considered is short enough so that the state of aging in the material (as indicated by \( a_t \) and \( a_T \)) remains approximately constant throughout the load. Graphically, log \( a_t \) and log \( a_T \) represent the distances (in log time) that the short-term response must be horizontally shifted to superpose it over the reference curve.

Isothermal physical aging occurs when the material is quenched from a temperature above \( T_q \) to a temperature \( T < T_q \) and held constant thereafter. In this case, the aging time \( t_e \) is simply defined as the time elapsed since the quench. For materials that have not reached thermodynamic equilibrium, the aging shift factor \( a_{t_e} \) function is well known as [28]

\[ a_{t_e}(t_e; T) = \left( \frac{t_{\text{ref}}}{t_e} \right)^{\mu(T)} \]

where \( \mu \) is the shift rate. For nonisothermal aging (varying temperature following quench), the shift factor \( a_{t_e} \) can no longer be described using a simple formula in terms of the time since quench. A number of studies have addressed the experimental approaches and mathematical models that can be used to determine/predict nonisothermal aging shift factor information in a consistent manner [28–33].

In this paper, it is assumed that the state of aging throughout a given thermal history is known beforehand, provided in terms of the combined shift factor \( a = a_{t_e} a_{T} \) determined using some appropriate method. The derivation below encapsulates the effects of nonisothermal physical aging in the calculation of the effective time \( \hat{\lambda} \) via this combined shift factor (see below). As such, the approach can also be employed to consider other effects modeled using horizontal shifting. For example, if a material response including the effects of physical aging, chemical aging and temperature satisfies the short-term relationship

\[ S(t_\text{ref}, \Omega, \hat{\lambda}) = S_{\text{ref}}(a_{t_e} a_{T} a_{T})|_{t_\text{ref}, \Omega_{\text{ref}}, T_{\text{ref}}} \]

where \( \Omega \) and its associated horizontal shift factor \( a_\Omega \) define the state of chemical aging in this material, then general laminate response of such a material can be analyzed using the method below by appropriately including \( a_\Omega \) in the corresponding effective time functions. Note, however, that aging and temperature effects tend to be strongly coupled when they occur together and cannot be considered independently of one another [e.g. \( a_{T} = f(t_e, \Omega, T) \) in Eq. (3)].

Finally, note that reference curves [for example, \( S_{\text{ref}} \) in Eq. (1)] always refer to momentary properties at a fixed aging and temperature state \( (t_{\text{ref}}, T_{\text{ref}}) \). Evolving aging and temperature effects are implemented through the use of shift factors and effective time theory. These modify the time variable of the reference curve, but never change the nature of the reference curve itself. Hence, although a material may respond in a complicated fashion to an involved thermal history, its reference curve(s) remain unchanged throughout.

3. General response using effective time theory for TRS materials

While the relationships in Eqs. (1)–(2) predict momentary response, they also provide the methodology
by which non-momentary response to varying loads may be predicted. Effective (reduced) time theory [5, 34–36] has been successfully employed to account for ongoing aging in long-term loading cycles [24, 28]. This approach uses the combined momentary shift factor \( a \) to create an alternative time scale for the solution of the response, by mapping the time since load initiation \( t \) into effective time \( \lambda(t) \) by the equation

\[
d\lambda = a(t)dt \rightarrow \lambda(t) = \int_0^t a(\xi)d\xi
\]  

(4)

Since the individual shift factors that make up \( a \) are always positive, \( \lambda(t) \) is a one-to-one mapping function of \( t \).

While physical aging leads to highly nonlinear behavior in glassy polymers, the material response can be reduced to linear viscoelastic by mapping the driving and response functions to the effective time domain. Such a mapping does not change the function values but only stretches and modifies the time axes to which they refer; for example, any stress point \((t_i, \sigma_i)\) in real time maps to the value \((\lambda_i, \sigma_i)\) in effective time. Consider the strain response of a material [compliance reference curve \( S_{ref}(t) \)] subjected to a constant stress \( \sigma_0 \) applied at \( t = 0 \). The strain response is found by the trivial mapping

\[
\sigma(t) \xrightarrow{\text{Map To } \lambda} \tilde{\sigma}(\lambda) = \sigma_0 S_{ref}(\lambda) \xrightarrow{\text{Map To } \epsilon(t)} \epsilon(t) = \sigma_0 S_{ref}[\lambda(t)]
\]  

(5)

where the \( \tilde{\cdot} \) notation indicates functions in the effective time domain. Note that if \( \lambda \) is such that the combined shift factor \( a = a_{ref} + a_t \) does not vary in the region \([0, \lambda]\) (the condition required for momentary response), the effective time is given by \( \lambda(t) = a_{ref} \lambda + a_t t \) and Eq. (5) becomes the momentary compliance \( S(t) = \epsilon(t)/\sigma_0 \) referred to in Eq. (1).

The relationship in Eq. (5) in the effective time domain can then be combined with the standard Boltzmann superposition principle [37] to determine an integral form for the strain response \( \epsilon(t) \) to an arbitrarily varying stress \( \sigma(t) \) as [34]²

\[
\sigma(t) \xrightarrow{\text{Map To } \lambda} \tilde{\sigma}(\lambda) = \int_0^{\lambda} S_{ref}(\lambda - \xi) \frac{d\tilde{\sigma}}{d\xi} d\xi \xrightarrow{\text{Evaluate } \tilde{\sigma}(\lambda) \text{ and Map To } \epsilon(t)} \epsilon(t)
\]  

(6)

²This form places any stress step occurring at time \( 0 \) inside of the integral, the derivative of which leads to a Dirac \( \delta \) function. Unless noted otherwise, integrals in this paper are intended in this manner in which the lower limit \( 0 \) can be equivalently replaced by \( 0^- \). Note in Eq. (5)–Eq. (6) that the stress function \( \sigma(t) \) is mapped to the effective time domain and inserted into the convolution integral; the strain function \( \epsilon(t) \) results from evaluating the integral.

Morland and Lee [34] demonstrated that Eq. (6) can alternatively be written as an integral in real time as

\[
\epsilon(t) = \int_0^t S_{ref}[\lambda(t) - \lambda(\xi)] \frac{d\sigma}{d\xi} d\xi
\]  

(7)

This form [Eq. (7)] is no longer a convolution integral, which eliminates many possible solution techniques. For this reason, this paper will perform general stress/strain predictions by mapping to effective time and using the convolution relationship of Eq. (6) for solution.

The linear viscoelasticity model used below requires that the material response can be both scaled and superposed in the effective time domain. For this condition to be generally satisfied, the state of aging in the material must be decoupled from the mechanical loading (i.e. the state of aging depends only on thermal history and is not affected by the applied mechanical loads). In this case, the effective time may be determined prior to the mechanical response solution by an appropriate method, so that at all desired solution times \( \lambda(\xi) \), the associated effective time \( \lambda(t) \) is known. Struik has demonstrated that for some cases, such as high stress loadings, the state of aging is affected by the mechanical loading [28]. If the stress–aging relationship can be satisfactorily modeled using effective time methods (i.e. via a modified \( a_{ref} \) or a new stress shift factor \( a_t \)), the model described below will also be applicable (although the effective time will need to be calculated during each solution step to account for the changing stress state).

4. Mechanical response of an aging fiber-reinforced lamina

The laminates considered below are constructed of a series of thin laminae (plies), stacked in a known sequence. The properties of the composite laminate will be determined using those of the individual laminae. Each lamina is modeled as a thin plate, and only the properties in the plane are considered. The fiber-reinforced lamina considered in this paper consists of parallel in-plane fibers, bonded together in a polymeric matrix material. For orientation, the lamina axial direction (1) is oriented along the fibers, the transverse direction (2) is perpendicular to (1) in the plane, and the shear direction (6) is shear in the 1–2 plane. The in-plane lamina response is governed by a matrix of four reference curve functions (compliance \( S \), modulus \( Q \)). For an aging lamina, these functions can be characterized by a series of isothermal physical aging tests at various aging times and temperatures. Methods used to determine behavior from such a test series have been described in several articles [22, 23, 28].

To determine the lamina strain \( \epsilon \) in terms of the stress \( \sigma \) and the compliance \( S \) in the most general case, an
understanding of four aging compliance responses must be known: fiber-direction compliance ($S_{11}$), fiber-transverse coupling compliance ($S_{12}$), transverse direction compliance ($S_{22}$) and shear compliance ($S_{66}$). Experimental studies have shown, however, that $S_{11}$ and $S_{12}$ exhibit little time dependence [38,39]; thus, the state of aging impacts these parameters only slightly. In this case, the analysis is simplified by approximating that all extension (non-shear) components map to the effective time associated with $S_{22}$ (exact in the case that $S_{11}$ and $S_{12}$ are constants). Shear behavior is governed by a different effective time associated with $S_{66}$. Defining the effective time in the transverse and shear directions as $\hat{\lambda}_2(t)$ and $\hat{\lambda}_6(t)$, respectively, the lamina strain vector $\varepsilon$ can be obtained in terms of the stress vector $\sigma$ and the lamina compliance matrix $S$ by four equations following the form of Eq. (6)

$$
\begin{bmatrix}
\sigma_1(t) \\
\sigma_2(t) \\
\sigma_6(t)
\end{bmatrix} 
\approx 
\begin{bmatrix}
\hat{\lambda}_2 \\
\hat{\lambda}_6
\end{bmatrix}
\begin{bmatrix}
S_{11}(\hat{\lambda}_2 - \xi) & S_{12}(\hat{\lambda}_2 - \xi) \\
S_{12}(\hat{\lambda}_2 - \xi) & S_{22}(\hat{\lambda}_2 - \xi)
\end{bmatrix}
\begin{bmatrix}
\tilde{\sigma}_1(\xi) \\
\tilde{\sigma}_2(\xi)
\end{bmatrix}
\mathrm{d}\xi 
\begin{bmatrix}
\varepsilon_1(t) \\
\varepsilon_2(t)
\end{bmatrix}
\begin{bmatrix}
\hat{\varepsilon}_1(\hat{\lambda}_2) \\
\hat{\varepsilon}_2(\hat{\lambda}_2)
\end{bmatrix} 
\Rightarrow 
\begin{bmatrix}
\tilde{\varepsilon}_1(\xi) \\
\tilde{\varepsilon}_2(\xi)
\end{bmatrix}
\begin{bmatrix}
\varepsilon_1(t) \\
\varepsilon_2(t)
\end{bmatrix}
$$

(8)

While the shift rate $\mu$ for amorphous polymers should not differ in tensile and shear loading [28], experimental studies of physical aging in composite laminates often find significant differences in $\hat{\lambda}_2(t)$ and $\hat{\lambda}_6(t)$ (typically manifested by different isothermal shift rates $\mu_{22}$ and $\mu_{66}$) [19–23]. Given these findings, it is useful to maintain the distinct effective time behavior in the shear and transverse directions in developing a general model of physical aging in composite laminates. The impact of differing shear and transverse aging behavior will be investigated in a later section.

Bradshaw and Brinson have shown (via inversion from $S$ to $Q$) that materials governed by Eq. (8) must correspond to modulus functions with the same effective time parameters [40,41]. Thus, the lamina stress vector $\sigma$ can be obtained in terms of the lamina strain vector $\varepsilon$ and the lamina modulus $Q$ as

$$
\begin{bmatrix}
\varepsilon_1(t) \\
\varepsilon_2(t) \\
\varepsilon_6(t)
\end{bmatrix} 
\approx 
\begin{bmatrix}
\hat{\lambda}_2 \\
\hat{\lambda}_6
\end{bmatrix}
\begin{bmatrix}
Q_{11}(\hat{\lambda}_2 - \xi) & Q_{12}(\hat{\lambda}_2 - \xi) \\
Q_{12}(\hat{\lambda}_2 - \xi) & Q_{22}(\hat{\lambda}_2 - \xi)
\end{bmatrix}
\begin{bmatrix}
\tilde{\varepsilon}_1(\xi) \\
\tilde{\varepsilon}_2(\xi)
\end{bmatrix}
\mathrm{d}\xi 
\begin{bmatrix}
\sigma_1(t) \\
\sigma_2(t) \\
\sigma_6(t)
\end{bmatrix}
\begin{bmatrix}
\hat{\sigma}_1(\hat{\lambda}_2) \\
\hat{\sigma}_2(\hat{\lambda}_2)
\end{bmatrix} 
\Rightarrow 
\begin{bmatrix}
\tilde{\sigma}_1(\xi) \\
\tilde{\sigma}_2(\xi)
\end{bmatrix}
\begin{bmatrix}
\sigma_1(t) \\
\sigma_2(t) \\
\sigma_6(t)
\end{bmatrix}
$$

(10)

Note that the lamina strain $\varepsilon$ in these expression is that due to load effects alone; other hygrothermal strains required to account for thermal expansion, moisture effects, etc., can be added to these expressions without incident and will be done so in a later section.

5. Reference curve functions

Efficient solution algorithms for solving Eqs. (8)–(11) exist when the reference curves are provided in terms of generalized Maxwell or generalized Voight models in terms of a series of relaxation/retardation times. In this derivation, both types of functions will be referred to as a Prony series [37]. While each material function in the lamina modulus $Q$ and compliance $S$ can have its own set of relaxation or retardation times, the analysis below is greatly simplified by the adoption of common relaxation/retardation values $\tau$. Since the coefficients of the Prony series are generally found by a mathematical fitting procedure [37], [41–43], it is usually straightforward to enforce this condition. The lamina modulus functions for this paper are given by

$$
Q(t) = 
\begin{bmatrix}
Q_{11}^0 & Q_{12}^0 & 0 \\
Q_{12}^0 & Q_{22}^0 & 0 \\
0 & 0 & Q_{66}^0
\end{bmatrix} + 
\sum_{p=1}^{P-1} 
\begin{bmatrix}
Q_{11}^{p} & Q_{12}^{p} & 0 \\
Q_{12}^{p} & Q_{22}^{p} & 0 \\
0 & 0 & Q_{66}^{p}
\end{bmatrix} e^{-\frac{t}{\tau_p}} 
$$

(12)

while the lamina compliance functions are given by

$$
S(t) = 
\begin{bmatrix}
S_{11}^0 & S_{12}^0 & 0 \\
S_{12}^0 & S_{22}^0 & 0 \\
0 & 0 & S_{66}^0
\end{bmatrix} + 
\sum_{p=1}^{P-1} 
\begin{bmatrix}
S_{11}^{p} & S_{12}^{p} & 0 \\
S_{12}^{p} & S_{22}^{p} & 0 \\
0 & 0 & S_{66}^{p}
\end{bmatrix} e^{-\frac{t}{\tau_p}} 
\left(1 - e^{-\frac{t}{\tau}}\right)
$$

(13)

The solution algorithm below is based on Prony series in the form in Eq. (12). To employ this algorithm for compliance, recast $S$ in Eq. (13) as

$$
S(t) = 
\begin{bmatrix}
S_{11}^0 & S_{12}^0 & 0 \\
S_{12}^0 & S_{22}^0 & 0 \\
0 & 0 & S_{66}^0
\end{bmatrix} + 
\sum_{p=1}^{P-1} 
\begin{bmatrix}
-S_{11}^{p} & -S_{12}^{p} & 0 \\
-S_{12}^{p} & -S_{22}^{p} & 0 \\
0 & 0 & -S_{66}^{p}
\end{bmatrix} e^{-\frac{t}{\tau_p}}; 
\quad S_{ij}^p = S_{ij}^0 + \sum_{p=1}^{P-1} S_{ij}^{p} 
$$

(14)

As mentioned previously, a typical simplification based on experimental data is that $S_{11}$ and $S_{12}$ are approximated as constant.
6. Taylor method for 1-D stress–strain prediction

To develop the solution method for Eqs. (8)–(11), consider first the general convolution integral equation

\[ y_0X(t) + \int_0^t X(t - \xi) \frac{dy}{d\xi}(\xi)d\xi = f(t) \]  

(15)

where \( X(t) \) is the material behavior function, \( y(t) \) is the driving function, and \( f(t) \) is the response function. The material is initially quiescent at \( t = 0 \) and the initial step in the driving function \( (y_0 \text{ at } t = 0) \) has been brought outside the integral. While many methods exist for analyzing such equations, one particularly powerful approach is the Taylor method [12], which employs a recursive algorithm to significantly reduce the number of calculations required for each solution step. Since the solution of Eq. (15) at any point is dependent upon the values at all previous points, this recursion avoids the enormous solution time and storage requirements that can occur for large problems.

The Taylor method is based upon two assumptions. First, the material response function is given by a Prony series of the form

\[ X(t) = X_0 + \sum_{p=1}^{P-1} X_p e^{-\frac{t}{\tau_p}} \]  

(16)

Second, after the initial step, the driving function \( y(t) \) is approximated as continuous piecewise linear between its definition points \( (t_i, y_i) \) (with \( t_0 = 0 \)). Under these conditions, the resulting solution can be obtained at the same time values as \( (t_i, f_i) \). The most impressive feature of the Taylor method is that a recursion allows the solution at the next time step \( t_{u+1} \) to be expressed solely in terms of results from the two previous solution times \( t_{u-1} \) and \( t_u \). Defining \( y_{u-1} = 0 \), the solution value \( f_u \) at any time step \( t_u (u \geq 0) \) is given by

\[ \mu_u(y_u - y_{u-1}) + \omega_u = f_u \]  

(17)

\[ \mu_u = X_0 + \sum_{p=1}^{P-1} X_p h^u_p; \quad \omega_u = X_0 y_{u-1} + \sum_{p=1}^{P-1} g^u_p \]

where \( g^u_p \) and \( h^u_p \) are given by\(^3\)

\[ g^u_p = \begin{cases} 0 & u = 0 \\ \left[ g^u_{p-1} + X_p h^u_{p-1}(y_{u-1} - y_{u-2}) \right] e^{-\frac{u-1}{\tau_p}} & u \geq 1 \end{cases} \]  

(18)

\[ h^u_p = \begin{cases} 1 & u = 0 \\ \tau_p \frac{X_p}{\tau_p - \tau_{u-1}} & u \geq 1 \end{cases} \]  

(19)

It is only necessary to know \( g^u_p \), \( h^u_p \), \( y_{u-2} \) and \( y_{u-1} \) to calculate the new values \( g^u_p \) and \( h^u_p \). Once these values are obtained, Eq. (17) can be used to obtain \( f_u \) from a known \( y_u \), or vice versa.

7. Lamina mechanical response using the Taylor method

In order to apply the Taylor method above to determine lamina response, the extension stress and strain functions \( (\varepsilon_1, \varepsilon_2, \sigma_1, \sigma_2) \) are first mapped to the transverse effective time space \( \lambda_2 \) and the shear stress and strain functions \( (\varepsilon_6, \sigma_6) \) are mapped to the shear effective time space \( \lambda_6 \); this is done at a series of discrete solution points \( t_u \) (with \( t_0 = 0 \)). The stress and strain vectors are only needed at these points, so define

\[ \varepsilon_i^u = \varepsilon_i(t_u) = \varepsilon_i(\lambda_2^u); \quad \sigma_i^u = \sigma_i(t_u) = \sigma_i(\lambda_6^u) \quad i = 1, 2, 6 \]  

(20)

where \( \lambda_2^1 = \lambda_2^u = \lambda_2(t_u) \) and \( \lambda_6^u = \lambda_6(t_u) \). With this notation, the lamina response at time \( t_u \) in Eqs. (8)–(11) can be combined into two systems as

\[ \sigma_i^u = \sum_{j=1, 2, 6} Q_{ij}(\lambda_2^u)\varepsilon_j^0 + \sum_{j=1, 2, 6} Q_{ij}(\lambda_6^u - \xi) \frac{d\varepsilon_j}{d\xi}(\xi)d\xi \]  

(21)

\[ \varepsilon_i^u = \sum_{j=1, 2, 6} S_{ij}(\lambda_2^u)\sigma_j^0 + \sum_{j=1, 2, 6} S_{ij}(\lambda_6^u - \xi) \frac{d\sigma_j}{d\xi}(\xi)d\xi \]  

(22)

Since the material functions \( (S, Q) \) are given by Prony series, the 1-D Taylor method can be used to solve each of these systems if the driving functions in the appropriate effective time domains \((\hat{\sigma}, \hat{\varepsilon})\) are assumed to be piecewise linear. This derivation is performed below.

7.1. Stress from strain and modulus

The 1-D Taylor method can be applied to the term in parentheses in Eq. (21); by extension, the solution for this expression becomes (for a specified \( i, j \))
Thus, either the stress $\sigma^u$ or strain $\varepsilon^u$ can be obtained by solution of this equation.

### 7.2. Strain from stress and compliance

Similarly, the 1-D Taylor method can be applied to the term in parentheses in Eq. (22); by extension, the solution for this expression becomes (for a specified $i, j$)

$$
\left( S_{ij}(\xi)\sigma^0_i + \int_0^{\xi_i} S_{ij}(\xi_i - \xi) \frac{d\tilde{\sigma}_j}{d\xi}(\xi)d\xi \right) = D_{ij}^{\mu}(\sigma_j^0 - \sigma_j^{u-1}) + \nu_{ij}^u
$$

$$
D_{ij}^{\mu} = S_{ij}^\infty + \sum_{p=1}^{P-1} (-S_{ij}^0) h_{ij}^p; \quad \nu_{ij}^u = S_{ij}^\infty \sigma_j^{u-1} + \sum_{p=1}^{P-1} \gamma_{ijp}^u
$$

where $h_{ij}^p$ is given by Eq. (25) and $\gamma_{ijp}^u$ is given by

$$
\gamma_{ijp}^u = \left\{ \begin{array}{ll}
0 & u = 0 \\
\gamma_{ijp}^{u-1} + (-S_{ij}^0) h_{ij}^p (\sigma_j^{u-1} - \sigma_j^{u-2}) & u \geq 1
\end{array} \right.
$$

Note that the use of $S_{ij}^\infty$ and $-S_{ij}^0$ is necessitated by the need to have a modulus-type Prony series [see Eq. (14)].

Substitution of Eq. (27) into (22) leads to the algebraic matrix-vector equation

$$
\varepsilon^u = D_{ij}^{\mu} \Delta \sigma^u + \nu^u
$$

$$
\nu_j^u = \sum_{i=1}^N \nu_{ij}^u; \quad \Delta \sigma_j^u = \sigma_j^u - \sigma_j^{u-1}
$$

As above, either the stress $\sigma^u$ or strain $\varepsilon^u$ can be obtained by solution of this equation.

### 7.3. Unification of compliance and modulus approaches

The application of classical lamination theory below is based upon the solution for the stress vector $\sigma^u$ in terms of the strain vector $\varepsilon^u$ as

$$
\sigma^u = L_{ij}^{\mu} \Delta \varepsilon^u + w^u
$$

For the case when modulus $Q$ is used to characterize the material, Eq. (26) is already in this form. For the case when compliance $S$ is used to characterize the material, Eq. (29) can be rearranged into this form with the substitutions

$$
L_{ij}^{\mu} = (D_{ij}^{\mu})^{-1}; \quad w^u = L_{ij}^{\mu} (\varepsilon^u - \nu^u) + \sigma^{u-1}
$$

Thus, the description of the relationship between lamina stress and strain at time $t_u$ is rendered independent of the form used as the basis for lamina description (compliance or modulus). For the remainder of the derivation, it will be assumed that during the evaluation at time $t_u$ matrix $L_{ij}^{\mu}$ and vector $w^u$ are known. This approach is similar to that used by Kennedy and Wang [17].

### 8. Hygrothermal strain effects

The preceding relationships were developed to describe the stress–strain response of an aging lamina. The strain in this case was that due mechanical response alone. Other strains may be induced in the lamina due to effects such as thermal expansion, volume recovery, moisture absorption, etc.; these are referred to as hygrothermal strains [3].

Since the above development was for a free-standing lamina (i.e., the lamina is not yet constrained by other differently oriented laminae), the total lamina strain can be determined by simply adding the hygrothermal and mechanical strains without affecting the mechanical response. To prevent the notation from becoming too complicated, define the total strain of the lamina as $\varphi$ in terms of the mechanical strain $\varepsilon$ and the hygrothermal strain $\psi$ such that
where $e_1$ and $e_2$ are the hygrothermal strains in the fiber and transverse directions, respectively. The above equation enforces the condition that hygrothermal effects do not lead to a shear strain component in a free-standing lamina, which would violate the symmetry condition of the lamina.

With this, the stress/strain response for the lamina from Eq. (30) can be written in terms of the total lamina strain as

$$\sigma_u = \mathbf{L}_u \Delta \varphi^u - \psi^u + w^u$$

(33)

$$\Delta \varphi^u = \begin{cases} \varphi^u_1 - \varphi^u_{1-1} \\ \varphi^u_2 - \varphi^u_{2-1} \\ \varphi^u_3 - \varphi^u_{3-1} \end{cases}; \psi^u = \mathbf{L}_u (\epsilon^u - \epsilon^{u-1})$$

In the model presented below, the total strain will be solved for. Once obtained, the mechanical strain $\epsilon^u$ will be recovered using the known hygrothermal strains as $\varphi^u - \epsilon^u$.

### 9. Stress/strain behavior an aging laminate

The expression in Eq. (33) incorporates the stress–strain response of a general aging lamina at time $t_u$ into a single algebraic equation; consideration of the different effective time behavior in the shear and transverse directions is contained in the parameters of this equation.

To assemble the overall laminate response, the expression for each lamina must be oriented with respect to the laminate axis. For the $n$th lamina in the laminate, the stress/strain response from Eq. (33) is given as

$$\sigma^u_n = \mathbf{L}^u_n \Delta \varphi^u_n - \psi^u_n + w^u_n$$

(34)

where $\mathbf{L}^u_n$, $w^u_n$, and $\psi^u_n$ are known constants (by evaluation using previous solution values). As shown in Fig. 1, define the angle between the fiber direction (1) of $n$th lamina and the laminate longitudinal axis ($x$) as $\theta_n$. The rotation matrix $\mathbf{T}_n$ for the $n$th lamina is then given by [1]

$$\mathbf{T}_n = \begin{bmatrix} \cos^2 \theta_n & \sin^2 \theta_n & 2 \sin \theta_n \cos \theta_n \\ -\sin \theta_n \cos \theta_n & \sin \theta_n \cos \theta_n & 0 \end{bmatrix}$$

(35)

The lamina stress ($\sigma^u_n$) and change in total strain ($\Delta \varphi^u_n$) vectors oriented along the laminate axis are needed to construct the laminate response; these are related to the fiber direction values by [1]

$$\sigma^u_n = \mathbf{T}_n \sigma^u; \Delta \varphi^u_n = \mathbf{T}_n^T \Delta \varphi^u$$

(36)

Substituting Eq. (36) into (34) and solving for $\sigma^u_n$ leads to

$$\sigma^u_n = \mathbf{P}_n^u \Delta \varphi^u_n + \mathbf{u}^u_n$$

(37)

Define the aging laminate force and moment vectors ($\mathbf{N}_n$, $\mathbf{M}_n$) at time $t_u$ in terms of the stress state in each ply as [1]

$$\mathbf{N}_n = \sum_{n=1}^{N} \int_{z_n}^{z_n+1} \sigma^u_n dz; \mathbf{M}_n = \sum_{n=1}^{N} \int_{z_n}^{z_n+1} \sigma^u_n dz$$

(40)

where the laminate is constructed of $N$ layers and $z_n$ is the lamina surface location defined in Fig. 1. Substitution of Eq. (39) into this expression and evaluating leads to the governing equation at time $t_u$ as

$$\mathbf{N}_n = \begin{bmatrix} \alpha^u_n \\ \beta^u_n \end{bmatrix} \Delta \varphi^u_n + \mathbf{a}^u_n$$

(41)

where the matrices $\alpha^u$, $\beta^u$, $\delta^u$ are given by

$$\alpha^u = \sum_{n=1}^{N} \mathbf{P}_n^u (z_n - z_{n-1})$$

(42)

$$\beta^u_n = \frac{1}{2} \sum_{n=1}^{N} \mathbf{P}_n^u (z_n - z_{n-1})^2$$

(43)
the \( a^u \) and \( b^v \) vectors are given by

\[
a^u = \sum_{n=1}^{N} u_n^u(z_n - z_{n-1})
\]

(45)

\[
b^v = \frac{1}{3} \sum_{n=1}^{N} u_n^v(z_n - z_{n-1})^2
\]

(46)

Based upon the known previous step information and the known hygrothermal strains at time \( t^t \), \( \alpha^u \), \( \beta^v \), \( \delta^u \), \( a^u \) and \( b^v \) can be assembled. Thus, Eq. (41) is a straightforward matrix equation that can be used to solve for the strain and curvature vectors \( (\varepsilon_{n}^{u}, \kappa_{n}) \) when supplied the laminate force and moment vectors \( (N, M) \), or vice versa.

Once the strain and curvature vectors \( (\varepsilon_{n}^{u}, \kappa_{n}) \) are known, the ply-level total strain \( \varphi_{n}^{u} \) can be recovered by evaluating Eq. (38) at \( z = \frac{1}{2}(z_n + z_{n-1}) \) and rotating it back to the lamina fiber direction [via Eq. (36)]. Subtraction of the hygrothermal strain \( \varepsilon_{n}^{u} \) leads to the mechanical strain \( \varepsilon_{n}^{u} \). The ply stress state \( (\sigma_{n}^{u}) \) can be determined using Eq. (34). Thus, all information about the response of the material is known at the current step and the procedure can be repeated for the next step.

The above procedure was incorporated into an ANSI C program which is capable of solving for the laminate displacement response \( (\varepsilon^{0}, \kappa) \) or the laminate load vectors \( (N, M) \) when the other is known. The known information is passed as data, and can take on any shape (i.e., the loading function is not restricted to constant or cyclic behavior). This program maintains the distinct aging behavior in the shear and transverse directions, permits coupled aging-temperature behavior via an appropriate effective time calculation, and allows for general time-dependent hygrothermal strains. It also supports the use of multiple material types in a given laminate; this is especially useful for constructing thick laminates that include a center honeycomb section or employ a change of material at some point in the laminate. Upon completion, the obtained laminate response is stored along with the individual ply information (stress, mechanical strain, total strain) that results during the solution.

10. Lamina material properties used in demonstration cases

In order to demonstrate the model presented above, the response of laminates under various conditions and layups are considered below. The material properties used in this study correspond to laminae constructed of continuous carbon fibers in an amorphous thermoplastic polyimide resin system (denoted IM7/K3B and
manufactured by DuPont). The glass transition temperature \( T_g \) of this material was measured to be 240°C. The isothermal and nonisothermal aging behavior of this material has been studied at Northwestern University and the material properties below were reported in a previous work [40]. The fiber-dominated compliance terms were approximated as constants \( S_{11} = 1/E_{11} \) and \( S_{12} = -v_{12}/E_{11} \), where \( E_{11} \) is the fiber-direction modulus of elasticity and \( v_{12} \) is the associated Poisson’s ratio; the actual values correspond to a similar material (denoted IM7/8320) at 200°C [19]. The time-dependent transverse \( (S_{22}) \) and shear \( (S_{66}) \) compliance functions were determined from isothermal physical aging tests at 208°C, 215°C and 225°C. Time-temperature superposition was then used to construct master reference curves. The sign control method [41] was used to fit \( S_{22} \) and \( S_{66} \) with 30 element Prony series [\( P = 30 \) in Eq. (13)]. The resulting compliance reference curves are shown in Fig. 2; these are shown extended approximately one decade beyond the experimental data studied. The related modulus functions [Eq. (12)] were then obtained via simplified lamina inversion technique [41] using the sign control method; the resulting modulus reference curves are shown in Fig. 3. The isothermal aging tests also provide the shift rate \( \mu \) and the temperature shift factor \( a_T \) for \( S_{22} \) and \( S_{66} \). These data points were fit with the appropriate curves shown in Figs. 4 and 5; the fitting curves were chosen to enforce the known behavior values \( (\mu = 0 \text{ at } T = T_g, a_T = 1 \text{ at } T = T_{ref}) \). These curves will be used to evaluate parameters for the subsequent studies. The predictions which follow use the modulus functions shown in Fig. 3 as the basis, although compliance functions can also be used with equivalent results.

11. Effect of differing shift rates on isothermal laminate response

As discussed previously, experimental results often show that the shift rate \( \mu \) differs for lamina response in the shear and transverse directions. As developed, the model here accounts for these different aging rates via separate effective time functions. Since this complicates the bookkeeping and analysis, the conditions under which a single average shift rate (and single effective time function) can be used are investigated for composite response prediction.
Several laminates of IM7/K3B were studied under the following conditions. First, the material was assumed to be in a state of isothermal physical aging at 225°C. Second, the average shift rate \( \mu_{\text{avg}} = \frac{1}{2}(\mu_2 + \mu_6) \) in each case was 0.80. Third, the material is initially quiescent (no mechanical or hygrothermal strains) when placed under a constant uniaxial stress \( \sigma_x = 6.9 \text{ MPa} \) (1000 psi) at an initial aging time \( t_0 = 1 \text{ h} \). Three cases for the values of the transverse \((\mu_2)\) and shear \((\mu_6)\) shift rates were considered: (A) \( \mu_2 = \mu_6 = 0.80 \), (B) \( \mu_2 = 0.9 \) and \( \mu_6 = 0.7 \), and (C) \( \mu_2 = 1 \) and \( \mu_6 = 0.6 \). Results such as those in case (B) are often obtained in experimental results, while those of case (C) would represent a highly unusual difference. From the isothermal shift rates, the effective times \( \lambda_2 \) and \( \lambda_6 \) are calculated using Eq. (4). The shift factor \( a = a_{\text{te}} \) is given by Eq. (2) where \( t_c(t) = t_0 + t \). These conditions provide all of the information necessary to make general laminate predictions.

For angle ply laminates (layup \([+\theta/\bar{\theta}]_s\)), large differences in response were obtained for the three cases above. To demonstrate this, the results for a \([+60/\bar{60}]_s\) laminate are shown in Fig. 6. Clearly the error in the prediction using the average shift rate is unacceptable for this case. Several other angle ply laminates were considered with similar results. One interesting point is that this study numerically verified that the reference shear compliance \( S_{66} \) can be obtained from a uniaxial test of a \([+45/-45]_s\) laminate (see the Appendix for analytical derivation); the obtained \( S_{66} \) was identical to the original lamina properties provided using the shear shift rate \( \mu_6 \) regardless of the value of \( \mu_2 \) or the presence of hygrothermal strains.

Incorporation of additional layers of a different fiber orientation drastically reduces both the resulting strain and the error in the above prediction; for example, the results for a \([+45/-45/90]_s\) laminate are shown below in Fig. 7. The strain results from a test of either a \([+45/-45]_s\) or a \([90]_s\) laminate would be very large (similar to Fig. 6 above), but the stiffness of a laminate is significantly increased with the multiple fiber directions, which in turn reduces the error due to the shift rate differences. This result was found for a number of different laminate configurations.

One other interesting result is to consider a more fiber-dominated layup than those above. In Fig. 8, the results for a quasi-isotropic \([+45/-45/90/0]_s\) laminate subjected to the same test are also shown. As in previous findings, neglecting the differing shift rate condition leads to relatively little error. Of greater interest is the fact that this laminate still demonstrates the time-dependent effects of physical aging despite the presence of the 0° ply.

Thus, it is reasonable to conclude that for layups with multiple fiber directions, differing shift rates in the shear and transverse directions can be accurately modeled with little error by employing a single average shift rate. For angle ply laminates, however, the differing shift rates need to be maintained to avoid significant errors.
12. Hygrothermal stress effects

In the preceding section, the strain response of a laminate due to a constant load undergoing isothermal physical aging was considered and hygrothermal strains were neglected. Here, for the same loading and initial aging time $t_{0}$, the effect of hygrothermal strains upon the ply stresses of each layer are considered below by examining the $[+45/−45/90]_{2}$ laminate at 225°C; results are compared to the ply stress values obtained in the previous section. The fiber direction hygrothermal strain $e_{1}$ was set to zero, while the transverse hygrothermal strain $e_{2}$ was determined using an experimentally obtained CTE value of $20 \times 10^{-6}/\text{C}$ with a value of $\Delta T = -15\text{°C}$ (assuming that the material has zero hygrothermal strains at $T = T_{g}$).

In applying the model above developed in this paper, the stresses in each ply (oriented with the ply fiber direction) are obtained. Fig. 9 shows the ply stress results for the $+45^\circ$ plies, while Fig. 10 shows the ply stress result the $90^\circ$ plies. At short times, the ply stresses are significantly different in each case. With elapsed time, however, the ply stresses converge towards a common value, demonstrating that residual stresses relax over time in a viscoelastic laminate; this result does not imply that the residual stresses will vanish as a very long time may be required for this process (especially for aging laminates).

The above findings show that it is important to consider and understand the hygrothermal strain state for accurate strength predictions based upon the ply-level stress state. Furthermore, the residual stress state will be time-dependent in an aging viscoelastic composite laminate. The model presented here rigorously incorporates hygrothermal strains and thus will accurately account for strains resulting from non-constant effects such as strains resulting from volumetric relaxation and thermal expansion under a varying temperature history.

13. Variable load nonisothermal aging laminate study

One application of the model developed here is to predict the mechanical response of a general composite laminate undergoing nonisothermal physical aging. This is possible provided that the effective times (which incorporate the nonisothermal physical aging effects) are known. This approach is demonstrated for several laminates below undergoing the uniaxial stress and temperature history shown in Fig. 11.

The temperature history above was chosen because it is similar to nonisothermal physical aging data studied at Northwestern University. Predictions of the non-isothermal shift factors $\sigma_{i}$ during this temperature history were made using the KAHR-$\sigma_{ic}$ model [40] using parameters developed from shear compliance data for IM7/K3B laminates [40]; the resulting $\sigma_{ic}$ values account for aging-temperature coupling effects [29,31–33] and are shown in Fig. 12. For comparison, the aging shift factor $\sigma_{ic}$ was also calculated directly from Eq. (2) ignoring the temperature coupling effects. The $\mu(T)$...
term in Eq. (2) causes the discontinuities seen in the approximate $a_{te}$ curves in Fig. 12. In both the KAHR-$a_{te}$ model and the approximate $a_{te}$ approach [using only Eq. (2), the aging shift factor is combined with the temperature shift factor as $a = a_{te}a_T$ for the effective time calculation used in the prediction. Note that in the approximate $a_{te}$ approach, this represents a simple superposition of aging and temperature effects. Comparison of the KAHR-$a_{te}$ model and the approximate method allow assessment of the importance of the coupled effects for composite laminate response.

The shape of the KAHR-$a_{te}$ prediction is as expected during this varied temperature history. In the temperature down-jumps, the shift factor smoothly decreases after the initial step. In the temperature up-jumps, the shift factor first increases and then decreases with the well-known shape obtained by other researchers for shift factor data [31] and volumetric relaxation [29]. Note that the KAHR-$a_{te}$ prediction is too high for the first temperature segment (which is isothermal) and the isothermal shift factor is instead used for this first segment in the KAHR-$a_{te}$ predictions.

The effective time for the cases studied below was found by integrating the aging shift factors ($a_{te}$) above multiplied by the temperature shift factors ($a_T$) from Fig. 5; this integration was performed using the trapezoidal rule. For the KAHR-$a_{te}$ predictions, the aging shift factors were assumed to be the same in both the shear and transverse directions (since no data is yet available for the transverse response parameters of the KAHR-$a_{te}$ model), but the different temperature shift factors in the shear and transverse directions were maintained.

In addition to the two nonisothermal aging viscoelastic analyses mentioned above, two additional simplifying assumptions were studied. In the first, the aging and temperature effects are completely neglected. In this case, the effective time in both the shear and transverse directions was simply set to the elapsed time since load application (ensures response is same as above cases immediately after load application but ignores accumulating aging effects). The other case of interest is the elastic solution, in which the modulus function $Q(t)$ is simply set to $Q(0)$. Thus, four cases are shown for each laminate below.

In summary, the strain response to the thermomechanical loading specified in Fig. 11 was considered for the following four cases:

- Elastic CLT—no viscoelastic or aging effects
- Viscoelastic—accumulating aging effects are neglected
- Viscoelastic + Aging—aging and temperature effects simply superposed
- Viscoelastic + Aging + KAHR-$a_{te}$—coupled aging and temperature effects

In each of the laminate predictions below, hygrothermal strains were also considered using the CTE value of $20 \times 10^{-6}/^\circ C$ found previously with the assumption that the hygrothermal strains are 0 at $T=T_g$. The fact that these hygrothermal strains are temperature dependent accounts for the fact that the minima and maxima of the elastic response cases below vary from cycle to cycle.

The first laminate studied is a [+60/-60]s angle ply. This leads to the strain response shown in Fig. 13. As in the isothermal case of the previous section, the laminate response is highly time-dependent. In this case, neglecting the ongoing aging effects leads to predictions that are too high by a factor of 3–10 once the first 10 h after the quench are passed. Conversely, the elastic analysis leads to strain predictions that are roughly 4–10 times too low at all but the shortest times. The KAHR-$a_{te}$ result and the approximate aging result are very similar until the second 225°C segment is reached; at that time, some significant differences occur.
As in the isothermal case, much less time-dependence is found when multiple fiber-directions are considered. For example, the second laminate studied is the \([+45/-45/90]_2\), which leads to the strain response shown in Fig. 14. As before, the fixed aging and elastic cases are too high and too low, respectively, but the errors are now much smaller. The predictions from the KAHR-aHe and superposition cases are now nearly indistinguishable. Note that the effect of the hygrothermal strains is clearly visible in Fig. 14, causing the jagged response of the curves at the temperature changes. Finally, it is also interesting to consider the response of the \([+45/-45/90/0]\) quasi-isotropic laminate as well. This leads to the response shown in Fig. 15, which is stiffer yet similar to the \([+45/-45/90]_2\) laminate.

14. Isothermal experimental result for \([+45/-45/90]_2\) laminate

Limited experimental testing of IM7/K3B \([+45/-45/90]_2\) laminates undergoing isothermal physical aging has been performed at Northwestern University at 225°C. Similar to the standard sequence test, the specimen was rejuvenated and then placed in a state of uniaxial stress at aging times \(t_a = 3/8, 3/4, 3/2, 3, 6, 12, 24\) and 48 h. The load was maintained for a duration equal to 10% of the aging time at the start of the load, and was then allowed to recover (no load) thereafter until the next load segment. The strain resulting from such a test was predicted using the model developed in this paper using the material properties in Figs. 2–5. In order to allow each load/unload segment to be clearly visible, the data and prediction are shown in the step space \(w(t)\) in Fig. 16; this approach maps each segment to a unit width [23]. The results show that the model captures the behavior fairly well. Two other predictions are considered: (1) assuming elastic response, and (2) a quasi-static viscoelastic approximation in which the strain response is assumed to be

\[ \epsilon(t) = \hat{S}[\dot{\lambda}(t)] \sigma(t) \]  

(47)

where \(\hat{S}[\dot{\lambda}(t)]\) is the long-term aging compliance of the laminate. This approximation retains the time-dependence properties of the composite but neglects hereditary effects [contrast Eqs. (8) and (47)]. The elastic case significantly under-predicts the data, while the quasi-static prediction significantly over-predicts the data. These results demonstrate that it is important to consider the effects of matrix viscoelasticity and physical aging in an appropriate fashion for accurate predictions.

15. Conclusions

A model was presented for predicting mechanical response of linear viscoelastic composite laminates. The model is capable of using either compliance or modulus...
response as its basis, and it permits differing aging behavior in the shear and transverse directions of a laminate. Time-varying hygrothermal strains and multiple materials system in a single laminate are also incorporated. The solution for the laminate response is obtained via a recursive algorithm (Taylor method) used to evaluate the governing convolution integral equations.

Several examples of long-term composite response were considered to investigate the effects of physical aging on composite laminate response. Material properties obtained for a polyamide matrix–carbon fiber laminate system were used as the basis of these calculations. The effect of differing shear and transverse shift rates on laminate response were investigated during uniaxial loading. For angle ply laminates, the difference in prediction was significant but for laminates with multiple fiber directions, fairly similar results were obtained using an average shift rate approach. The effect of hygrothermal strains was also considered. As expected, the ply stresses caused by hygrothermal strains were initially large but decreased with time due to the viscoelastic nature of the laminate. Finally, a case with nonisothermal aging and variable loading was demonstrated using two different techniques for calculating the aging state of the laminate. The results demonstrated that the effects of physical aging must be considered for accurate predictions involving angle ply laminates. For multi-directional laminates, aging impacts the results but to a much lesser degree. In the latter case, a simplified superposition approach which neglects aging-temperature coupling is sufficient for accurate material prediction under the nonisothermal condition studied. These results indicate that for many composite laminates under certain nonisothermal aging conditions, complex coupling effects can be safely ignored in the analysis of viscoelastic aging effects. In all cases, it was demonstrated that for load histories within 20–30°C of \( T_g \), viscoelastic and aging effects must be considered; both the elastic and quasi-static approximations are inadequate.

Appendix. Aging shear compliance from a 45° angle ply laminate

In the body of this paper, a general solution method for an aging laminate subjected to an arbitrary stress–strain loading was developed. One special case of interest is the constant uniaxial load test of an aging 45° angle ply laminate (such as a \([\pm 45]_s\)), which is often used as the test basis to determine the shear compliance of an individual lamina. It will be shown below that the shear compliance obtained from such a test exactly recovers the shear behavior of the individual lamina even if the aging behavior differs in the shear and transverse directions.

Before deriving the aging case, it is useful to recall the results for an elastic \([\pm 45]_s\) angle ply composite laminate. If this laminate is subjected to a uniaxial stress \( \sigma_0 \), the elastic shear compliance \( S_{66} \) is obtained as

\[
S_{66} = \frac{1}{Q_{66}} = 2 \frac{\epsilon_{s0}^0 - \epsilon_{t0}^0}{\sigma_0} \tag{A1}
\]

where \( \epsilon_{s0}^0 \) and \( \epsilon_{t0}^0 \) are the observed mid-plane strains in the directions with and perpendicular to the applied stress, respectively. This result is one well-known method for determining the shear compliance of an elastic lamina [3]. A similar relationship can be derived for an aging viscoelastic laminate, even in the general case when the aging behavior differs in the shear and transverse directions.

Consider the laminate strain that results from a uniaxial test of a \([\pm 45] \), aging viscoelastic laminate: while the extensional strains \( \epsilon_{s0}^0 \) and \( \epsilon_{t0}^0 \) will be general and time-dependent, the mid-plane shear strain should always be zero to satisfy symmetry conditions. Suppose that \( \epsilon_{s0}^0 \) and \( \epsilon_{t0}^0 \) are known from a uniaxial test; the strain state rotated to the lamina axis orientation for both the +45 and −45 laminae using Eq. (36) with the transformation matrix \( T \) from Eq. (35) is obtained as

\[
\begin{bmatrix}
\psi_1(t) \\
\psi_2(t) \\
\psi_0(t)
\end{bmatrix}_{\pm 45} = \begin{bmatrix}
U(t) \\
U(t) \\
-cV(t)
\end{bmatrix}
\]

\[
u(t) = \frac{\epsilon_{s0}^0(t) + \epsilon_{t0}^0(t)}{2} ; \quad V(t) = \epsilon_{s0}^0(t) - \epsilon_{t0}^0(t) ; \quad c = \text{sign}(\theta)
\]

where \( \varphi \) is the total strain (oriented with the fiber direction) existing in each lamina during the test in question. The total strain consists of strain due to the applied load (\( \epsilon \)) and hygrothermal effects (\( e \)). Assuming that the hygrothermal strains are independent of the lamina position in the laminate, the mechanical strain in each layer can be calculated as

\[
\begin{bmatrix}
\epsilon_1(t) \\
\epsilon_2(t) \\
\epsilon_0(t)
\end{bmatrix}_{\pm 45} = \begin{bmatrix}
\varphi_1(t) \\
\varphi_2(t) \\
\varphi_0(t)
\end{bmatrix}_{\pm 45} - \begin{bmatrix}
\epsilon_1(t) \\
\epsilon_2(t) \\
0
\end{bmatrix}_{\pm 45} = \begin{bmatrix}
U(t) - \epsilon_1(t) \\
U(t) - \epsilon_2(t) \\
-cV(t)
\end{bmatrix}
\]

The ply-level stress response of the lamina (in the lamina orientation) due to this applied strain function

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4This implies that the hygrothermal strain in each layer is identical. This assumption may not be acceptable in some cases (e.g., transient moisture absorption). In such cases, the earlier assumption that the overall laminate shear strain \( \epsilon_{s0}^0 \) will be invalid, necessitating a different set of initial assumptions for the derivation.
can be obtained for a generally aging lamina following
the form of Eqs. (10)–(11) recast in the real time form of
Eq. (7) as
\[
\begin{pmatrix}
\sigma_1(t) \\
\sigma_2(t) \\
\sigma_0(t)
\end{pmatrix}_{\pm 45} =
\begin{pmatrix}
f_1(t) + f_2(t) \\
f_3(t) + f_4(t) \\
-cf_5(t)
\end{pmatrix}
\] (A4)

where
\[
\begin{pmatrix}
f_1(t) \\
f_2(t) \\
f_3(t) \\
f_4(t) \\
f_5(t)
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
\int_0^\lambda \frac{d}{dt} \frac{d^2}{d\xi^2} \frac{dV}{d\xi} d\xi
\end{pmatrix}
\] (A5)

This stress state for each lamina can now be rotated
back to the laminate frame by again using Eq. (35)–(36)
and the sign parameter c previously described as (note
\(c^2 = 1\))
\[
\begin{pmatrix}
\sigma_x(t) \\
\sigma_y(t) \\
\sigma_{xy}(t)
\end{pmatrix}_{\pm 45} = \begin{pmatrix}
f_1(t) + f_2(t) + f_3(t) + f_4(t) + 2fs_0(t) \\
f_1(t) + f_2(t) + f_3(t) + f_4(t) - 2fs_0(t) \\
c[f_1(t) + f_2(t) - f_3(t) - f_4(t)]
\end{pmatrix}
\] (A6)

The laminate force vector is obtained by the stress
above over the laminate thickness [see Eq. (40)]. Calling
the thickness of each lamina as \(dz/4\), the force vector of the
\([\pm 45]\), laminate divided by the laminate thickness
(which is known to be uniaxial with magnitude \(\sigma_0\)) becomes
\[
\begin{pmatrix}
N_x(t) \\
N_y(t) \\
N_{xy}(t)
\end{pmatrix} = \begin{pmatrix}
\sigma_0 \\
0 \\
0
\end{pmatrix}
\] (A7)

As in the elastic case, subtract \(\sigma_f\) from \(\sigma_x\) in Eq. (A7) to find
\[
\sigma_0 = 2fs_0(t) = 2 \int_0^\lambda Q_{66}[\lambda_0 - \lambda_0(\xi)] \frac{dV(\xi)}{d\xi} d\xi
\] (A8)

In determining this expression, the differing effective
time behavior in the shear and transverse directions has
been maintained; hence, there is no approximation used
for the aging behavior in the above relationship.

To demonstrate the relationship between Eq. (A8)
and the shear compliance \(S_{66}(t)\), map the strain function
\(V(t)\) to the shear effective time domain \(\lambda_0\) as \(V(\lambda_0)\);
Eq. (A8) can be written as [see Eq. (6)–(7)]
\[
\sigma_0 = 2 \int_0^\lambda Q_{66}[\lambda_0 - \xi] \frac{dV(\xi)}{d\xi} d\xi
\] (A9)

The Laplace transform of this equation is given by
\[
\frac{\sigma_0}{s} = 2Q_{66}(s)[sV(s) - V(0^-)]
\] (A10)

The relationship between \(Q_{66}\) and \(S_{66}\) in the Laplace
domain is given by
\[
Q_{66}(s)S_{66}(s) = \frac{1}{s^2}
\] (A11)

Substitution of this expression, recognition that the
material was in the unstrained state prior to application
of the uniaxial load, and taking the inverse Laplace
transform to the shear effective time \(\lambda_0\) domain leads to
\[
2 \frac{\tilde{V}(\lambda_0)}{\sigma_0} = S_{66}(\lambda_0)
\] (A12)

which can be mapped back to real time as
\[
\varepsilon_{eff}(t) = \frac{S_{66}(\lambda_0)}{\sigma_0}; \quad \varepsilon_{eff}(t) = 2[\varepsilon_s(t) - \varepsilon_f(t)]
\] (A13)

where \(\varepsilon_{eff}(t)\) is the effective shear strain. This name is
given since a \([\pm 45]\), laminate under uniaxial stress \(\sigma_0\) leads to \(\varepsilon_{eff}(t)\) identical to the shear strain that would be
found for a single lamina under shear stress \(\sigma_0\). Note
that the shear compliance \(S_{66}\) must be left in terms of
the shear effective time function \(\lambda_0(t)\); this is because the
shear compliance \(S_{66}\) is a reference curve defined in the
absence of aging effects.

The shear compliance expression in Eq. (A13) was
derived exactly, and did not rely on approximations of
the aging behavior (such as averaging the effective time
behavior) in the formulation. Hence, the \([\pm 45]\) angle
ply composite laminate specimen tested in uniaxial
stress can accurately return the aging shear behavior of
the individual lamina. Note that this analysis can be
simplified to the case of linear viscoelasticity by simply
setting the effective time to time in both the shear and
transverse directions (\(\lambda_2 = \lambda_6 = t\). Thus, Eq. (A13) also
exactly describes the shear compliance for a linear vis-
coelastic lamina in the absence of physical aging.
References